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BRIOT–BOUQUET SUBORDINATION PROPERTIES FOR ANALYTIC FUNCTIONS INVOLVING CHOI–SAIGO–SRIVASTAVA INTEGRAL OPERATOR

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Abstract

We consider a new subclass of starlike functions involving the Choi–Saigo–Srivastava integral operator associated with the sine function in open unit disc. In view of this function class, we examine majorization properties and Briot–Bouquet differential subordination relations for such functions.

Key words: starlike function, sine function, Choi–Saigo–Srivastava integral operator, Briot–Bouquet differential subordination, majorization

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1. Introduction. Denote by $\mathcal{A}$ the class of analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Denote by Ω the class of Schwarz functions w which are analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$. For analytic functions f_1 and f_2 in $\mathbb{D}$, we state that f_1 is subordinate to f_2, symbolized by $f_1 \prec f_2$, if there exists a function w in Ω satisfying $f_1(z) = f_2(w(z))$. The comprehensive details of subordination can be found in [1].

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The convolution or Hadamard product of the functions $f_1(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $f_2(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is defined by

$$f_1(z) * f_2(z) = (f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in \mathbb{D}).$$

Majorization is an important subject for investigating geometric properties of analytic functions. In 1967, MacGregor [2] introduced the concept of majorization. If f_1 and f_2 are two analytic functions in $\mathbb{D}$, then we say that f_1 is majorized by f_2 in $\mathbb{D}$ denoted by

$$f_1(z) \ll f_2(z), \quad (z \in \mathbb{D})$$

if there exists an analytic function φ in $\mathbb{D}$ satisfying

$$|\varphi(z)| \leq 1 \text{ and } f_1(z) = \varphi(z) f_2(z), \quad (z \in \mathbb{D}). \tag{1.2}$$

The Briot–Bouquet differential subordination is given by

$$\varphi(z) + \frac{z\varphi'(z)}{\eta\varphi(z) + \mu} \prec \phi(z), \quad (\eta, \mu \in \mathbb{C}, \eta \neq 0) \tag{1.3}$$

with $\varphi(0) = \phi(0) = 1$. If the univalent function $q(z) = 1 + q_1 z + q_2 z^2 + \cdots$ has the property that $\varphi \prec q$ for all analytic functions φ satisfying (1.3), then it is called a dominant of (1.3). If $\tilde{q} \prec q$ for all dominants q of (1.3), then a dominant $\tilde{q}$ is said to be the best dominant of the differential subordination (see [3,4]).

Ma and Minda [5] investigated the class of analytic functions ϕ with positive real part in $\mathbb{D}$ that map the disc $\mathbb{D}$ onto regions starlike with respect to 1, symmetric with respect to the real axis and normalized by the conditions $\phi(0) = 1$ and $\phi'(0) > 0$, and introduced the starlike functions class by

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z), \quad z \in \mathbb{D} \right\}.$$

For the case $\phi(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$, the family of Janowski starlike functions $\mathcal{S}^*[A, B]$ is obtained (see [6]). When $A = 1 - 2\delta$ $(0 \leq \delta < 1)$ and $B = -1$, we have the family $\mathcal{S}^*(\delta)$ of starlike functions of order δ. Particularly, $\delta = 0$ yields the usual class $\mathcal{S}^*(0) = \mathcal{S}^*$ of starlike functions. Recently, various authors have introduced and studied several Ma–Minda type classes of starlike functions [7–9]. In [10], Cho et al. considered the class $\mathcal{S}^*_{\sin}$ of starlike associated with the sine function if the subordination inclusion $zf'(z)/f(z) \prec \phi(z) = 1 + \sin z$ holds with $\phi(0) = 1$ and $\operatorname{Re} \phi(z) > 0$ for all $z \in \mathbb{D}$.

In 1984, Carlson and Shaffer [11] defined a linear operator $\mathcal{L}(a,c) : \mathcal{A} \to \mathcal{A}$ given by

$$\mathcal{L}(a,c)f(z) = \varphi(a,c;z) * f(z), \tag{1.4}$$

where the incomplete beta function $\varphi(a,c;z)$ is defined by

$$\varphi(a,c;z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n, \quad (a \in \mathbb{R}, c \in \mathbb{R}\backslash\mathbb{Z}_0^-, \mathbb{Z}_0^- = \{0,-1,\ldots\})$$

and the Pochhammer symbol $(\sigma)_n$ is defined by

$$(\sigma)_n = \frac{\Gamma(\sigma+n)}{\Gamma(\sigma)} = \begin{cases} 1, & \text{if } n = 0 \\ \sigma(\sigma+1)\cdots(\sigma+n-1), & \text{if } n \in \mathbb{N}. \end{cases}$$

The Carlson–Schaffer operator given by (1.4) contains the Ruscheweyh differential operator [12];

$$L(\rho+1,1)f(z) = D^{\rho}f(z) := \frac{z}{(1-z)^{\rho+1}} * f(z), \quad (\rho \in \mathbb{R}, \rho > -1).$$

Motivated by the Ruscheweyh operator, Choi, Saigo and Srivastava [13] defined the operator $\mathcal{I}_{\rho,\gamma} : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{I}_{\rho,\gamma}f(z) = f_{\rho,\gamma}(z) * f(z), \quad (\rho > -1, \gamma > 0), \tag{1.5}$$

where

$$\frac{z}{(1-z)^{\rho+1}} * f_{\rho,\gamma}(z) = \frac{z}{(1-z)^{\gamma}}.$$

It is clear that for $\gamma = 2$, this operator reduces to the Noor integral operator defined in [14]. Also, note that $\mathcal{I}_{0,2}f(z) = zf'(z)$ and $\mathcal{I}_{1,2}f(z) = f(z)$.

Using (1.5), we have the following recursive relation for the Choi–Saigo–Srivastava integral operator:

$$z(\mathcal{I}_{\rho,\gamma}f(z))' = \gamma\mathcal{I}_{\rho,\gamma+1}f(z)\imath(\gamma-1)\mathcal{I}_{\rho,\gamma}f(z). \tag{1.6}$$

In view of the Choi–Saigo–Srivastava integral operator $\mathcal{I}_{\rho,\gamma}$, we define a new Ma–Minda type starlike function class associated with the sine function.

Definition 1.1. A function $f \in \mathcal{A}$ given by (1.1) belongs to the class $\mathcal{S}_{\rho,\gamma}(\sin)$ if it satisfies the condition

$$\frac{z(\mathcal{I}_{\rho,\gamma}f(z))'}{\mathcal{I}_{\rho,\gamma}f(z)} \prec \phi(z) := 1 + \sin z, \tag{1.7}$$

where $\mathcal{I}_{\rho,\gamma}$ is given by (1.5) and $z \in \mathbb{D}$.

In this paper, we define a new class of Ma–Minda type starlike function involving the Choi–Saigo–Srivastava integral operator associated with the sine function denoted by $\mathcal{S}_{\rho,\gamma}(\sin)$. We first find majorization property for this function class. We also employ a different method based on the Briot–Bouquet differential subordination which was investigated by Miller and Mocanu [4], and establish several new Briot–Bouquet differential subordination results for the operator $\mathcal{I}_{\rho,\gamma}$.

2. Majorization properties. We first begin to examine majorization property for the class $\mathcal{S}_{\rho,\mu}(\sin)$. For proving the result, we need the following lemma given by NEHARI ([15], p. 168).

Lemma 2.1. *If φ is analytic and bounded in $\mathbb{D}$, then*

$$|\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2},$$

where $|z|<1$.

Theorem 2.2. *Let $f \in \mathcal{A}$ and suppose that $g \in \mathcal{S}_{\rho,\gamma}(\sin)$. If $\mathcal{I}_{\rho,\gamma}f(z) \ll \mathcal{I}_{\rho,\gamma}g(z)$ for all $z \in \mathbb{D}$, then for $|z| \leq r_1$, we have*

$$|\mathcal{I}_{\rho,\gamma+1}f(z)| \leq |\mathcal{I}_{\rho,\gamma+1}g(z)|,$$

where r_1 is the smallest positive root of the equation

$$(1-r^2)(\gamma-\sinh r)-2r=0.$$

Proof. Since $g \in \mathcal{S}_{\rho,\gamma}(\sin)$, then from (1.7) we write

$$\frac{z(\mathcal{I}_{\rho,\gamma}g(z))'}{\mathcal{I}_{\rho,\gamma}g(z)} = 1+\sin w(z), \tag{2.1}$$

where $w(z)=c_1z+c_2z^2+\cdots$ is bounded and analytic in $\mathbb{D}$ satisfying

$$w(0)=0 \quad \text{and} \quad |w(z)| \leq |z|. \tag{2.2}$$

By using (1.6) into (2.1) and in view of the conditions in (2.2), we obtain

$$|\mathcal{I}_{\rho,\gamma}g(z)| \leq \frac{\gamma}{\gamma-|\sin z|}|\mathcal{I}_{\rho,\gamma+1}g(z)|. \tag{2.3}$$

Since $\mathcal{I}_{\rho,\gamma}f(z)$ is majorized by $\mathcal{I}_{\rho,\gamma}g(z)$, in view of (1.2), we write

$$\mathcal{I}_{\rho,\gamma}f(z) = \varphi(z)\mathcal{I}_{\rho,\gamma}g(z). \tag{2.4}$$

Differentiating on both sides of equation (2.4) with respect to z, and routine calculations gives

$$z(\mathcal{I}_{\rho,\gamma}f(z))' = z\varphi'(z)\mathcal{I}_{\rho,\gamma}g(z)+z\varphi(z)(\mathcal{I}_{\rho,\gamma}g(z))'. \tag{2.5}$$

By using (1.6) into (2.5) together with (2.4), we get

$$\mathcal{I}_{\rho,\gamma+1}f(z) = \frac{1}{\gamma}z\varphi'(z)\mathcal{I}_{\rho,\gamma}g(z)+\varphi(z)\mathcal{I}_{\rho,\gamma+1}g(z). \tag{2.6}$$

Using Lemma 2.1, and substituting (2.3) into (2.6), we get

$$|\mathcal{I}_{\rho,\gamma+1}f(z)| \leq \left[\frac{|z|(1-|\varphi(z)|^2)}{(1-|z|^2)}\frac{1}{\gamma-|\sin z|}+\varphi(z)\right]|\mathcal{I}_{\rho,\gamma+1}g(z)|. \tag{2.7}$$

Taking $|z|=r$ and $|\varphi(z)|=\epsilon$ $(0\leq\epsilon\leq1)$, and using the inequality $|\sin z|\leq \sinh r$ given by Cho et al. ([10], p. 5) into (2.7), we get

$$|\mathcal{I}_{\rho,\gamma+1}f(z)| \leq \Theta(r,\epsilon)|\mathcal{I}_{\rho,\gamma+1}g(z)|,$$

where

$$\Theta(r,\epsilon)=\frac{r(1-\epsilon^2)}{(1-r^2)}\frac{1}{\gamma-\sinh r}+\epsilon.$$

In order to determine r_1, we choose

$$\begin{aligned} r_1 &= \max\{r\in(0,1):\Theta(r,\epsilon)\leq 1, \forall\epsilon\in[0,1]\}\\ &= \max\{r\in(0,1):\nu(r,\epsilon)\geq 0, \forall\epsilon\in[0,1]\}, \end{aligned}$$

where

$$\nu(r,\epsilon)=(1-r^2)(\gamma-\sinh r)-r(1+\epsilon).$$

Since $\dfrac{\partial}{\partial\epsilon}\nu(r,\epsilon)=-r<0$, therefore $\nu(r,\epsilon)$ takes its minimum for $\epsilon=1$, namely

$$\min\{\nu(r,\epsilon)\geq 0, \epsilon\in[0,1]\}=\nu(r,1):=\nu(r),$$

where

$$\nu(r)=(1-r^2)(\gamma-\sinh r)-2r.$$

Moreover, since $\nu(0)=\gamma>0$, and $\nu(1)=-2<0$, thus there exists r_1 such that $\nu(r)\geq 0$ for all $r\in[0,r_1]$, where r_1 is the smallest positive root. This completes the proof. □

3. Subordination properties. In this section, we investigate the Briot–Bouquet differential subordination properties. To prove our main results, we need to give the following lemmas.

Lemma 3.1 ([16]). *Let ϕ $(\phi(0)=1)$ be convex univalent in $\mathbb{D}$, and let φ of the form $\varphi(z)=1+b_1z+b_2z^2+\cdots$ $(\varphi(0)=1)$ be analytic in $\mathbb{D}$. If*

$$\varphi(z)+\frac{1}{\mu}z\varphi'(z)\prec\phi(z), \quad (\mu\neq 0,\ \operatorname{Re}\mu\geq 0)$$

then

$$\varphi(z)\prec\tilde{\phi}(z)=\frac{\mu}{z^\mu}\int_0^z t^{\mu-1}\phi(t)dt\prec\phi(z), \tag{3.1}$$

and $\tilde{\phi}$ is the best dominant of equation (3.1).

Lemma 3.2 ([3], Theorem 2)**.** *Let η ($\eta \neq 0$) and μ be complex constants, and let ϕ ($\phi(0) = 1$) be convex univalent function in $\mathbb{D}$ with* $\mathrm{Re}(\eta\phi(z) + \mu) > 0$. *Let φ be analytic in $\mathbb{D}$ and satisfy the subordination* (1.3)*. If the Briot–Bouquet differential equation*

$$q(z) + \frac{zq'(z)}{\eta q(z) + \mu} = \phi(z), \quad (q(0) = 1) \tag{3.2}$$

has a univalent solution q, then

$$\varphi(z) \prec q(z) \prec \phi(z),$$

and q is the best dominant of (1.3)*.*

The differential equation (3.2) *has a formal solution given by*

$$q(z) = z^{\mu}[H(z)]^{\eta}\left(\eta \int_0^z [H(t)]^{\eta} t^{\mu-1} dt\right)^{-1} - \mu/\eta, \tag{3.3}$$

where

$$H(z) = z \exp \int_0^z \frac{\phi(t) - 1}{t} dt.$$

In the first theorem, we get Briot–Bouquet differential subordination for the operator $\mathcal{I}_{\rho,\gamma}$.

Theorem 3.3. *Let $\lambda > 0$ and $\zeta \geq 1$. If a function $f \in \mathcal{A}$ satisfies the condition*

$$(1 - \lambda)\frac{\mathcal{I}_{\rho,\gamma} f(z)}{z} + \lambda \frac{\mathcal{I}_{\rho,\gamma+1} f(z)}{z} \prec 1 + \sin z, \tag{3.4}$$

then

$$\mathrm{Re}\left\{\left(\frac{\mathcal{I}_{\rho,\gamma} f(z)}{z}\right)^{1/\zeta}\right\} > \left(\frac{\gamma}{\lambda}\int_0^1 u^{\frac{\gamma}{\lambda}-1}(1 - \sinh u)du\right)^{1/\zeta}. \tag{3.5}$$

The result is sharp.

Proof. Consider the analytic function

$$\varphi(z) = \frac{\mathcal{I}_{\rho,\gamma} f(z)}{z}, \qquad (z \in \mathbb{D})$$

with $\varphi(0) = 1$ in $\mathbb{D}$. Differentiating the above equality and using equation (1.6), we get

$$\frac{\mathcal{I}_{\rho,\gamma+1} f(z)}{z} = \varphi(z) + \frac{1}{\gamma} z\varphi'(z),$$

and by applying the subordination condition (3.4), we arrive at

$$(1-\lambda)\frac{\mathcal{I}_{\rho,\gamma}f(z)}{z}+\lambda\frac{\mathcal{I}_{\rho,\gamma+1}f(z)}{z}=\varphi(z)+\frac{\lambda}{\gamma}z\varphi'(z)\prec 1+\sin z.$$

By using Lemma 3.1 on the right-hand side of the above equation, we obtain

$$\varphi(z)\prec\frac{\gamma}{\lambda}z^{-\gamma/\lambda}\int_0^z t^{\frac{\gamma}{\lambda}-1}(1+\sin t)dt,$$

or

$$\frac{\mathcal{I}_{\rho,\gamma}f(z)}{z}=\frac{\gamma}{\lambda}\int_0^1 u^{\frac{\gamma}{\lambda}-1}(1+\sin uw(z))du, \tag{3.6}$$

where w is a Schwarz function. Due to Cho et al. ([10], p. 5), we conclude that

$$\operatorname{Re}\big(1+\sin uw(z)\big)>1-\sinh ur,\quad |z|=r\in(0,1).$$

Taking real part of both sides and letting $r\to 1^-$, from equation (3.6), we arrive at

$$\operatorname{Re}\left(\frac{\mathcal{I}_{\rho,\gamma}f(z)}{z}\right)>\frac{\gamma}{\lambda}\int_0^1 u^{\frac{\gamma}{\lambda}-1}(1-\sinh u)du>0,\quad (z\in\mathbb{D}) \tag{3.7}$$

where $\lambda>0$. Since $\operatorname{Re}(w^{1/\zeta})\geq\operatorname{Re}(w)^{1/\zeta}$ for $\operatorname{Re}(w)>0$ and $\zeta\geq 1$, from inequality (3.7) we obtain the inequality (3.5). To prove sharpness, we take $f\in\mathcal{A}$ defined by

$$\frac{\mathcal{I}_{\rho,\gamma}f(z)}{z}=\frac{\gamma}{\lambda}\int_0^1 u^{\frac{\gamma}{\lambda}-1}(1+\sinh uz)du.$$

For this function we find that

$$(1-\lambda)\frac{\mathcal{I}_{\rho,\gamma}f(z)}{z}+\lambda\frac{\mathcal{I}_{\rho,\gamma+1}f(z)}{z}=1+\sin z,$$

and

$$\frac{\mathcal{I}_{\rho,\gamma}f(z)}{z}\to\frac{\gamma}{\lambda}\int_0^1 u^{\frac{\gamma}{\lambda}-1}(1-\sinh u)du,$$

as $z\to 1^-$. Thus, the proof is completed. □

In the next theorem, we find a univalent solution of the Briot–Bouquet differential equation, and observe that this solution is the best possible of the Briot–Bouquet differential subordination for the class $\mathcal{S}_{\rho,\mu}(\sin)$.

Theorem 3.4. *If a function f belongs to the class $\mathcal{S}_{\rho,\mu}(\sin)$ such that $\mathcal{I}_{\rho,\gamma}f(z) \neq 0$, $\gamma > 0$, and*

$$\mathrm{Re}\left(\sin z + \gamma\right) > 0, \quad (z \in \mathbb{D})$$

then

$$\frac{z\big(\mathcal{I}_{\rho,\gamma}f(z)\big)'}{\mathcal{I}_{\rho,\gamma}f(z)} \prec q(z) \prec 1 + \sin z, \tag{3.8}$$

where

$$q(z) = z^{\gamma} e^{Si(z)} \left(\int_0^z t^{\gamma-1} e^{Si(t)} dt \right)^{-1} - \gamma + 1, \tag{3.9}$$

and q is the best dominant of (3.8).

Proof. Consider the analytic function

$$\varphi(z) = \frac{z(\mathcal{I}_{\rho,\gamma}f(z))'}{\mathcal{I}_{\rho,\gamma}f(z)}, \qquad (z \in \mathbb{D})$$

with $\varphi(0) = 1$. By using equation (1.6), we obtain

$$\gamma \frac{\mathcal{I}_{\rho,\gamma+1}f(z)}{\mathcal{I}_{\rho,\gamma}f(z)} = \varphi(z) + \gamma - 1,$$

and logarithmic differentiation with respect to z together with routine calculations give

$$\frac{z(\mathcal{I}_{\rho,\gamma+1}f(z))'}{\mathcal{I}_{\rho,\gamma+1}f(z)} = \varphi(z) + \frac{z\varphi'(z)}{\varphi(z) + \gamma - 1} \prec \phi(z) = 1 + \sin z. \tag{3.10}$$

Now, let us consider the Briot–Bouquet differential equation

$$q(z) + \frac{zq'(z)}{q(z) + \gamma - 1} = \phi(z) = 1 + \sin z, \tag{3.11}$$

where q ($q(0) = 1$) is analytic and $\phi(z) = 1 + \sin z$ is convex univalent with $\phi(0) = 1$ in $\mathbb{D}$, and let $P(z) = \eta\phi(z) + \mu$. In view of equation (3.11) and Lemma 3.2, we observe that $\eta = 1$, $\mu = \gamma - 1$ and

$$P(z) = \sin z + \gamma.$$

For proving $\mathrm{Re}(P(z)) > 0$, it is enough to set $z = e^{it}$, $t \in [0, \pi]$ under the condition $\gamma > 0$. Furthermore, $P(z)$ and $1/P(z)$ are convex. Hence, there is a univalent solution of the equation (3.11) and we get this solution by using the steps given in Lemma 3.2. Since $\phi(z) = 1 + \sin z$, in view of Lemma 3.2 we find

$$H(z) = z \exp \int\limits_0^z \frac{\phi(t) - 1}{t} dt \quad = z \exp \int\limits_0^z \frac{\sin t}{t} dt = z e^{\mathrm{Si}(z)},$$

where $\mathrm{Si}(z)$ is the Sin integral. Setting this result together with $\eta = 1$ and $\mu = \gamma - 1$ into the formula (3.3), we obtain equation (3.9) which is the univalent solution of equation (3.11). Since φ is analytic and satisfies (3.10), then we derive

$$\varphi(z) \prec q(z) \prec \phi(z) =: 1 + \sin z$$

and, q is the best dominant of equation (3.8). □

REFERENCES

[1] DUREN P. L. (1983) Univalent functions, Grundlehren Math. Wiss., vol. **259** [Fundamental Principles of Mathematical Sciences], New York, Springer-Verlag.

[2] MACGREGOR T. H. (1967) Majorization by univalent functions, Duke Math. J., **34**, 95–102.

[3] EENIGENBURG P., S. S. MILLER, P. MOCANU, M. READE (1983) On a Briot–Bouquet differential subordination, Internat. Schriftenreihe Numer. Math., vol. **64** [International Series of Numerical Mathematics], Basel, Birkhäuser Verlag, 339–348.

[4] MILLER S. S., P. MOCANU (1985) Univalent solutions of Briot–Bouquet differential equations, J. Differential Equations, **56**(3), 297–309.

[5] MA W. C., D. MINDA (1992) A unified treatment of some special classes of univalent functions. In: Proceedings of the Conference on Complex Analysis, Tianjin (eds Z. Li, F. Ren, L. Yang and S. Zhang), New York, International Press, 157–169.

[6] JANOWSKI W. (1973) Some extremal problems for certain families of analytic functions I, Ann. Polon. Math., **28**, 297–326.

[7] KUMAR V., N. E. CHO, V. RAVICHANDRAN, H. M. SRIVASTAVA (2019) Sharp coefficient bounds for starlike functions associated with the Bell numbers, Math. Slovaca, **69**(5), 1053–1064.

[8] MENDIRETTA R., S. NAGPAL, V. RAVICHANDRAN (2014) On a subclass of strongly starlike functions asscociated with exponential function, Bull. Malays. Math. Sci. Soc., **38**(1), 365–386, DOI: 10.1007/s40840-014-0026-8.

[9] ALOTAIBI A., M. ARIF, M. A. ALGHAMDI, S. HUSSAIN (2020) Starlikness associated with Cosine Hyperbolic function, Mathematics, **8**(7), 1118, `https://doi.org/10.3390/math8071118`.

[10] CHO N. E., V. KUMAR, S. S. KUMAR, V. RAVICHANDRAN (2019) Radius problems for starlike functions associated with the sine function, Bull. Iranian Math. Soc., **45**(1), 213–232.

[11] CARLSON B. C., D. B. SHAFFER (1984) Starlike and prestarlike hypergeometric functions, SIAM J. Math. Anal., **15**(4), 737–745.

[12] Ruscheweyh S. (1975) New criteria for univalent functions, Proc. Amer. Math. Soc., **49**, 109–115.
[13] Choi J. H., M. Saigo, H. M. Srivastava (2002) Some inclusion properties of a certain family of integral operators, J. Math. Anal. Appl., **276**(1), 432–445.
[14] Noor K. I. (1999) On new classes of integral operators, J. Nat. Geom., **16**(1–2), 71–80.
[15] Nehari Z. (1952) Conformal mappings, New York-Toronto-London, McGraw-Hill Book Co., Inc.
[16] Hallenbeck D. J., S. Ruscheweyh (1975) Subordination by convex functions, Proc. Amer. Math. Soc., **52**, 191–195.

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